10. VAINBERG M.M. and TRENOGIN V.A., Theory of the Bifurcation of Solutions of Non-linear Equations. Nauka, Moscow, 1969.

Translated by M.D.F.

PMM U.S.S.R.,Vol.49,No.4.pp.482-489,1985
Printed in Great Britian

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ASYMPTOTIC FORM OF THE STRESS INTENSITY COEFFICIENTS IN QUASISTATIC TEMPERATURE PROBLEMS FOR A DOMAIN WITH A CUT*

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Plane quasistatic thermoelasticity problems are investigated for domains of arbitrary shape with a cut in the case of an instataneous temperature change on the boundary. The asymptotic form of the stresses is investigated in the neighbourhood of a crack tip.

Certain quasistatic temperatue problems were solved earlier in /1-5/ (see /6/ also) for the development of cracks on parts of whose surfaces a constant temperature occurs at the initial instant and is maintained. Expressions are obtained for the stress intensity coefficients at the crack tip.

Quasistationary thermoelasticity problems are investigated below for domains with cut in a more general asymptotic sense. A plane domain with a cut whose boundary is instantaneously cooled or heated is examined in Sects.1–3. Since the shape of the domain contour can be arbitrary, it is impossible to speak of the explicit solution of the thermoelasticity boundary value problem. Nevertheless, an expression is successfully found for the principal terms of the asymptotic form of the stress intensity coefficients at the most dangerous initial times (from the viewpoint of crack propagation). In particular, the asymptotic form of the fracture time is determined as a function of the temperature jump at the crack tip.

Note that the principal term of the tensile stress intensity coefficient is independent of the contour shape, and agrees with the intensity coefficient of the same problem for a plane with a cut.

Analogous results are obtained in Sec.4 for the problem of an instantaneous change in the endface temperature of a thin plate from whose side surfaces heat is transferred to the external medium, where the stress intensity coefficients found are explicitly expressed in terms of those in the absence of heat transfer. This enables an asymptotic analysis to be made of the stresses near a crack tip at the initial times.

The results obtained in this paper emerge from the asymptotic solution of the heat conduction equations as $t \to 0$ for a domain with a cut and the method proposed in /7/ for calculating the stress intensity coefficients.

1. Formulation of the boundary value problems. To be specific we will examine plane strain. As is well-known, the plane state of stress with zero heat transfer from the external medium is realized on replacing the Lamé constant λ by $\lambda_* = 2\lambda\mu(\lambda + 2\mu)$, and γ by $\gamma_* = (1-2\nu)\gamma/(1-\nu)$, where $\gamma = 2\mu\alpha_T(1+\nu)/(1-2\nu)$; μ is the shear modulus, α_T is the coefficient of linear expansion, and ν is Poisson's ratio, in which connection, only the appropriate constants vary inthe asymptotic formulas indicated later.

Let Ω_0 be a plane domain with a smooth boundary Γ_0 (see the Figure). There is a rectilinear cut of length l in Ω_0 that connects the origin $O \in \Omega_0$ with the point $A \in \Gamma_0$. We denote the upper and lower edges of the cut by l_+ and l_-. We understand Γ to be the contour Γ_0 supplemented with two drawn segments l and Ω to be the domain bounded by Γ Let Ω^c be the closure of the domain Ω in the sense of its internal metric. To simplify the discussion, we will consider the angle formed by the contour Γ_0 and the segment l to be a right angle, and the contour Γ_0 itself to be rectilinear near the point A.

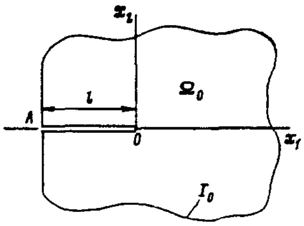
The temperature T is determined from the solution of the boundary value problem

*Prikl.Matem.Mekhan.,49,4,627-636,1985

$$\partial T/\partial t - \Delta T = 0 \quad \text{on} \quad \Omega \times (0, \infty) \tag{1.1}$$
$$T = 0 \quad \text{on} \quad \Gamma \times (0, \infty), \; T|_{t=0} = T_0$$

The thermal diffusivity is thereby assumed to be equal to one, which obviously does not restrict the generality.

The displacement vector $\mathbf{u}$ generated by this temperature field is found from the solution of the following boundary value problem ($\mathbf{n}, \boldsymbol{\tau}$ are the normal and tangent to Γ):



$$\mu\Delta\mathbf{u} + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \gamma\,\mathrm{grad}\,T \quad \text{on} \quad \Omega \tag{1.2}$$
$$\lambda\,\mathrm{div}\,\mathbf{u} + 2\mu\,\partial u_n/\partial n = \gamma T \quad \text{on} \quad \Gamma$$
$$\mu\,(\partial u_n/\partial \tau + \partial u_\tau/\partial n) = 0 \quad \text{on} \quad \Gamma$$

2. Asymptotic form of the temperature as $t \to +0$.

The solutions of three model problems are required to describe the singularities in the temperature T on Γ as $t \to +0$, related to the presence of corner points $0, A$.

1°. Selfsimilar solutions for a plane with a cut, quadrant, and half-plane. Let L be the solution of the homogeneous heat conduction equation in the domain $\{(x, t){:}\ r > 0,\ |\theta| < \pi,\ t > 0\}$, where $x = (x_1, x_2)$ and (r, θ) are the polar coordinates of the point x. The function L is subjected to boundary and initial conditions: $L|_{\theta=\pm\pi} = 0$, $L|_{t=0} = 1$. We shall seek L in the form $L = l(\rho, \theta)$, where $\rho = r^2/(4t)$, and we obtain the following boundary value problem for l:

$$\left(\rho^2 \frac{\partial}{\partial\rho} + \left(\rho\frac{\partial}{\partial\rho}\right)^2 + \frac{1}{4}\frac{\partial^2}{\partial\theta^2}\right) l = 0, \quad l|_{\theta=\pm\pi} = 0, \quad l \underset{\rho\to\infty}{\longrightarrow} 1 \tag{2.1}$$

Let $u_j(\theta) = \pi^{-1/2}\sin{}^1/_2 j(\theta + \pi)$ $(j = 1, 2, \ldots)$ be eigenfunctions of the operator $d^2/d\theta^2$ in the segment $[-\pi, \pi]$ with Dirichlet conditions on its ends. Keeping in mind the Fourier series in the system of functions $\{u_j\}$ for the ones, it is natural to represent l in the form of the series

$$l(\rho, \theta) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\, l_j(\rho)\cos\left(j + \frac{1}{2}\right)\theta \tag{2.2}$$

Satisfying the boundary value problem, we obtain

$$l_j(\rho) = \frac{\Gamma(5/4 - j/2)}{\Gamma(3/2 + j)}\,\rho^{1/4 + j/2} e^{-\rho}\,\Phi\left(\frac{5}{4} + \frac{j}{2};\ \frac{3}{2} + j;\ \rho\right)$$

where Φ is the degenerate hypergeometric function /8/. The asymptotic formula

$$l(\rho, \theta) = \frac{4}{\pi}\frac{\Gamma(5/4)}{\Gamma(3/2)}\,\rho^{1/4}\cos\frac{\theta}{2} + O(\rho^{3/4}), \quad \rho \to +0 \tag{2.3}$$

hence follows.

It is more convenient to find the asymptotic form of the function l for large ρ directly from the boundary value problem (2.1). Let χ be a smooth function on the positive half-axis that equals one in $[0, {}^1/_4]$ and zero in $[{}^1/_2, \infty)$. We shall seek the asymptotic form of the function l in the form

$$l(\rho, \theta) = 1 - \chi(\tau)\, g(\rho\sin^2\tau) + q(\rho, \theta), \quad \tau = \pi - |\theta| \tag{2.4}$$

Substituting (2.4) into (2.1) and integrating, we obtain that $g(z) = \mathrm{erfc}\sqrt{z}$.

The function q is the solution of a boundary value problem analogous to (2.1), where $O(e^{-b\rho})$ is on the right side of the equation in place of zero (b is a certain positive number). It can be shown by expanding $q(\rho, \theta)$ in a trigonometric series in $\sin{}^1/_2 k(\theta + \pi)$ or by using energy estimates that q decreases as $\rho \to \infty$ more rapidly than any power of ρ (the lengthy proof of this fact is omitted).

The selfsimilar solution $M(r^2/(4t), \theta)$ is constructed analogously for the quadrant $Q = \{x{:}\ r > 0,\ 0 < \theta < \pi/2\}$, i.e., the solution of the homogeneous heat conduction equation in $Q \times (0, \infty)$ that satisfies the conditions $M|_{\theta=0,\pi/2} = 0$, $M|_{t=0} = 1$. The explicit form of the function M is not used. It is useful just to keep in mind the asymptotic formulas

$$M(\rho, \theta) = c\rho\sin 2\theta + O(\rho^{3/2}), \quad \rho \to 0$$
$$M(\rho, \theta) = 1 - \chi(\tau)\,\mathrm{erfc}(\rho^{1/2}\sin\tau) + O(\rho^{-N}), \quad \rho \to \infty$$

where χ is the same shearing function as before, $\tau = \min\{\theta, \pi/2 - \theta\}$ and N is any positive number.

The solution $G(x, t) = \operatorname{erf}(x_1/(2t^{1/2}))$ of the homogeneous heat conduction equation in the domain $\{(x, t)\colon t > 0,\ x_1 > 0\}$ that satisfies the conditions $G|_{x_1=0} = 0$, $G|_{t=0} = 1$ is later also required.

2°. *Local estimate.* To estimate the residuals occuring on replacing T by the selfsimilar solutions constrйcted above, we prove the following assertion about the local estimate for the solutions of the heat conduction equation.

Lemma. Let U, V be domains in the plane $\{x\}$, $\bar{U} \subset V$ and R a function from $C([0, t_0]; W_2^1(\Omega)) \cap C^1([0, t_0]; W_2^{-1}(\Omega))$ that satisfies the initial boundary value problem

$$(\partial/\partial t - \Delta) R = f \quad \text{on} \quad \{(x, t)\colon 0 < t < t_0,\ x \in \Omega \cap V\} \tag{2.5}$$
$$R|_{t=0} = 0 \quad \text{on} \quad \Omega \cap V,\ R = 0 \quad \text{on} \quad \{(x, t)\colon 0 < t < t_0,\ x \in \Gamma \cap V\}$$

Then there exists a postive constant c dependent on U, V such that

$$\| R(\cdot, t)\|^p_{L_p(\Omega\cap U)} \leqslant c\int_0^t (\| f(\cdot, \tau)\|^p_{L_p(\Omega\cap V)} + \| R(\cdot, \tau)\|^p_{L_p(\Omega\cap V)})\, d\tau \qquad (\forall p \in [1, \infty)) \tag{2.6}$$

Proof. Let $F \in C^2(R^1)$ be a non-negative even function such that $F(0) = 0$, $0 \leqslant F'' \leqslant c$. We also introduce the non-negative function $\eta \in C_0^\infty(V \cap \Omega^c)$ that equals one in $U \cap \Omega$. Multiplying (2.5) by $\eta F'(R)$ and integrating by parts, we find

$$\frac{d}{dt}\int_\Omega \eta F(R)\, dx + \int_\Omega \eta F''(R)\,|\nabla_x R|^2\, dx - \int_\Omega F(R)\,\Delta\eta\, dx = \int_\Omega f F'(R)\,\eta\, dx$$

therefore

$$\int_\Omega \eta F(R)\, dx \leqslant c\int_0^t\int_\Omega (|f|^p\eta + |F'(R)|^{p/(p-1)}\eta + F(R)\,|\Delta\eta|)\, dx\, d\tau$$

Let $F = (R^2 + \delta^2)^{p/2} - \delta^p$, $\delta > 0$ for $1 \leqslant p \leqslant 2$. When $p > 2$ we set

$$F(R) = |R|^p \quad \text{for} \quad |R| < T;\quad F(R) = 2^{-1}p(p-1)\,T^{p-2}\,|R|^2 - p(p-2)\,T^{p-1}\,|R| + 2^{-1}(p-1)(p-2)\,T^p \quad \text{for} \quad |R| > T$$

Passing to the limit as $\delta \to 0$, $T \to \infty$, we obtain the estimate (2.6).

We note that for $f = 0$ the solution of problem (2.5) allows of the following estimate on $[0, t_0] \times (\Omega \cap V)$ (it is obtained from (2.6) by induction over N):

$$\| R(\cdot, t)\|^p_{L_p(\Omega\cap U)} \leqslant c_N t^{pN}\int_0^t \| R(\cdot, \tau)\|^p_{L_p(\Omega\cap U)}\, d\tau, \quad N = 0, 1, \ldots \tag{2.7}$$

3°. *Asymptotic form of the function T.* Let δ be a small positive number $U_\delta(O) = \{x\colon 0 < r < \delta,\ |\theta| < \pi\}$. By virtue of the estimate (2.7) the following inequality holds:

$$\| T(\cdot, t) - T_0 L(\cdot, t)\|_{L_p(U_\delta(O))} \leqslant c_N T_0 t^N, \quad 1 \leqslant p < \infty$$
$$N = 0, 1, \ldots$$

where L is the function defined in Sec. 1°. The following inequality, used later, is therefore obtained:

$$\int_\Omega r^{-3}\,|T(x, t) - T_0 L(x, t)|\, dx \leqslant ct^N \tag{2.8}$$

Let (r, θ) be polar coordinates with centre the point A and $U_\delta(A) = \{x\colon 0 < r < \delta,\ 0 < |\theta| < \pi/2\}$.

Again applying the estimate (2.7) to $R = T - T_0 M$, we find that

$$\| T(\cdot, t) - T_0 M(\cdot, t)\|_{L_1(U_\delta(A))} \leqslant c_N T_0 t^N \tag{2.9}$$

Here M is the selfsimilar solution determined in Sec. 1° for the first quadrant, that is continued in a clear manner into the fourth quadrant.

Let P now be any point of the contour Γ such that

$$|P - A| > \delta,\ |P - O| > \delta;\ U_\delta(P) = \{x \in \Omega\colon |x - P| < \delta\}.$$

We introduce the coordinates (n, s) in $U_\delta(P)$ where n is the distance to Γ, $|s|$ is the distance from the nearest point to x on the contour Γ to P, and the sign of s is selected in conformity with the positive direction of traversing the contour. The Laplace operator in (n, s) coordinates has the form

$$\zeta^{-1}\left(\frac{\partial}{\partial n}\zeta\frac{\partial}{\partial n}+\frac{\partial}{\partial s}\zeta^{-1}\frac{\partial}{\partial s}\right),\quad \zeta=1-nk(s)$$

(k is the curvature). Consequently

$$\left(\frac{\partial}{\partial t}-\Delta\right)\operatorname{ert}\left(\frac{n}{2t^{1/2}}\right)=-\frac{k(s)}{\zeta(\pi t)^{1/2}}\exp\left(-\frac{n^2}{4t}\right)$$

In view of the maximum principle, for the heat-conduction equation

$$\left|T-T_0\operatorname{erf}\left(\frac{n}{2t^{1/2}}\right)\right|\leqslant T_0$$

Moreover, the boundary and initial values of the function T, $T_0\operatorname{erf}(n/2t^{1/2})$ agree in $U_\delta(P)$; consequently, on the basis of inequality (2.6) we obtain

$$\left\|\left(T-T_0\operatorname{erf}\left(\frac{n}{2t^{1/2}}\right)\right\|_{L_1(U_\delta(P))}\leqslant c\left(\int_0^t\tau^{-1/2}\int_{U_\delta(P)}\exp\left(-\frac{n^2}{4\tau}\right)dn\,ds\,d\tau+t\right)\leqslant ct \tag{2.10}$$

And finally, let Q be any point of the domain Ω such that the distance between it and the boundary Γ is greater than δ. Let $U_\delta(Q)$ be a circle of radius δ with centre the point Q.

By virtue of estimate (2.7)

$$\|T(\cdot,t)-T_0\|_{L_1(U_\delta(Q))}\leqslant c_N T_0 t^N,\quad N=0,1,\ldots \tag{2.11}$$

3. Asymptotic form of the stress intensity coefficients as $t\to+0$. 1°. *Asymptotic form of displacements near the crack tip.* We consider the boundary value problem (1.2) in which the time t enters as a parameter. If $t>0$, then the quantities grad T and T have weak singularities at the crack tip, and consequently, the asymptotic form of the solution of this problem is

$$(u_r,u_\theta)(r,\theta)=c_1(\cos\theta,-\sin\theta)+c_2(\sin\theta,\cos\theta)+$$
$$(4\mu)^{-1}\sqrt{r/2\pi}\,(K_\mathrm{I}\varphi^{(\mathrm{I})}(\theta)+K_\mathrm{II}\varphi^{(\mathrm{II})}(\theta))+O(r),\quad r\to 0$$
$$\varphi^{(\mathrm{I})}(\theta)=((2\varkappa-1)\cos\theta/2-\cos3\theta/2,\ -(2\varkappa+1)\sin\theta/2-\sin3\theta/2)$$
$$\varphi^{(\mathrm{II})}(\theta)=((2\varkappa-1)\sin\theta/2-3\sin3\theta/2,\ (2\varkappa-1)\cos\theta/2-3\cos3\theta/2)$$

Here u_r, u_θ are components of the displacement vector in a polar coordinate system, c_1, c_2 are certain functions of time, K_I, K_II are stress intensity coefficients dependent on t. $\varkappa=3-4\nu$ for plane strain and $\varkappa=(3-\nu)/(\nu+1)$ for the plane state of stress.

Following /7/, we describe the procedure for calculating the coefficients K_I and K_II. Let $\mathbf{z}^{(\mathrm{I})}$ and $\mathbf{z}^{(\mathrm{II})}$ denote the displacement fields satisfying the homogeneous Lamé equations and the boundary conditions $\sigma(\mathbf{z}^{(j)})\cdot\mathbf{n}=0$ on Γ, bounded outside any neighbourhood of the point O and having the asymptotic form

$$(z_r^{(j)},z_\theta^{(j)})(r,\theta)=[2(1+\varkappa)(2\pi r)^{1/2}]^{-1}\psi^{(j)}(\theta)+O(1),\quad r\to 0$$
$$\psi^{(\mathrm{I})}(\theta)=((2\varkappa-1)\cos3\theta/2-3\cos\theta/2,\ -(2\varkappa-1)\sin3\theta/2-3\sin\theta/2)$$
$$\psi^{(\mathrm{II})}(\theta)=((2\varkappa+1)\sin3\theta/2-\sin\theta/2,\ (2\varkappa-1)\cos3\theta/2-\cos\theta/2)$$

Since $T=0$ on $\Gamma\setminus(0,\infty)$ and $\sigma_n(U)=\sigma_{n\tau}(U)=0$ on Γ, then according to /7/ for $t>0$

$$K_j(t)=\gamma\int_\Omega\operatorname{grad}T(x,t)\,\mathbf{z}^{(j)}(x)\,dx,\quad j=\mathrm{I},\mathrm{II} \tag{3.1}$$

Integrating by parts in (3.1), we obtain

$$K_j(t)=-\gamma\int_\Omega T(x,t)\,h_j(x)\,dx=-\gamma\int_\Omega(T(x,t)-T_0)\,h_j(x)\,dx \tag{3.2}$$
$$h_j(x)=\operatorname{div}\mathbf{z}^{(j)}(x)$$

The following equation was used here

$$\int_\Gamma z_n^{(j)}\,d\Gamma=0$$

which follows from the Betti formula for the vectors $\mathbf{z}^{(j)}$, x. We note that $h_j = h_j^{(0)} + O(r^{-1/4})$, where

$$h_{\mathrm{I,\,II}} = \frac{(1-\varkappa)}{(1+\varkappa)\sqrt{2\pi}}\, r^{-1/2} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \frac{3\theta}{2}$$

2°. *Asymptotic form of the stress intensity coefficients. Theorem* 1. The asymptotic formulas

$$K_{\mathrm{I}}(t) = \frac{4}{\pi}\Gamma\left(\frac{3}{4}\right)\mu m T_0 t^{1/4} + \frac{2\gamma T_0}{\pi^{1/2}} t^{1/2} \int_\Gamma h_{\mathrm{I}}(x)\, d\Gamma + O(t^{3/4}) \tag{3.3}$$

$$K_{\mathrm{II}}(t) = \frac{2\gamma T_0}{\pi^{1/2}} t^{1/2} \int_\Gamma h_{\mathrm{II}}(x)\, d\Gamma + O(t^{3/4}), \quad t \to +0$$

hold, where $m = \alpha_T (1+\nu)(1-\nu)^{-1}$, $\gamma = 2\mu\alpha_T (1+\nu)(1-2\nu)^{-1}$ for plane strain and $m = \alpha_T(1+\nu)$, $\gamma = 2\mu\alpha_T(1+\nu)(1-\nu)^{-1}$ for the plane state of stress with zero heat transfer from the external medium.

The integrals in (3.3) are understood in the principal value sense:

$$\int_\Gamma h_j(x)\, d\Gamma = \lim_{\varepsilon\to 0} \int_{C(\varepsilon)} h_j(x)\, d\Gamma, \quad C(\varepsilon) = \{x \in \Gamma\colon |x - 0| \geqslant \varepsilon\}$$

The limit on the right exists because

$$\left(\int_{C(\varepsilon_1)} - \int_{C(\varepsilon_2)}\right) h_j^{(0)}\, d\Gamma = 0$$

Proof. We fix a sufficiently small number $\delta > 0$. Let $\Gamma_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \Gamma) < \delta\}$, $U_\delta(O)$ and $U_\delta(A)$ are neighbourhoods of the points O and A defined in Sec. 1.2, $\Gamma_{0\delta} = \Gamma_\delta \setminus (U_\delta(O) \cup U_\delta(A))$. We set

$$I_{0j}(t) = -\int_{U_\delta(O)} (L-1)\, h_j\, dx, \quad I_{Aj}(t) = -\int_{U_\delta(A)} (M-1)\, h_j\, dx \tag{3.4}$$

$$I_{1j}(t) = \int_{\Gamma_{0\delta}} \operatorname{erfc}\left(\frac{n}{2t^{1/2}}\right) h_j(x)\, dx$$

where n is the distance to the boundary. Let

$$R_j(t) = K_j(t) - \gamma T_0 (I_{0j}(t) - I_{Aj}(t) + I_{1j}(t))$$

It follows from (3.2) and (3.4) that

$$R_j(t) = -\gamma\Bigg(\int_{U_\delta(O)} (T - T_0 L)\, h_j\, dx - \int_{U_\delta(A)} (T - T_0 M)\, h_j\, dx -$$

$$\int_{\Gamma_{0\delta}} \left(T - T_0 \operatorname{erfc}\left(\frac{n}{2t^{1/2}}\right)\right) h_j\, dx - \int_{\Omega - \Gamma_\delta} (T - T_0)\, h_j\, dx\Bigg)$$

Since $h_j = O(r^{-1/2})$ then by using the estimates (2.8)-(2.11), we obtain that $R_j(t) = O(t)$.

Therefore, to obtain (3.3) it is sufficient to investigate the functions I_{0j}, I_{Aj}, I_{1j} for small t and we will therefore do this.

We evaluate the integrals

$$I_{0j}(t) = -\int_{U_\delta(O)} (L-1)\, h_j^{(0)}\, dx$$

We have

$$I_{01}(t) = -\int_0^\delta \int_{-\pi}^{\pi} \frac{4}{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{2k+1}\left(l_k\left(\frac{r^2}{4t}\right) - 1\right)\cos\left(k + \frac{1}{2}\right)\theta\, \frac{(1-\varkappa)}{(1+\varkappa)\sqrt{2\pi}} \times \tag{3.5}$$

$$r^{-1/2}\cos\frac{3}{2}\theta\, dr\, d\theta = \frac{2(1-\varkappa)}{3\sqrt{\pi}(1-\varkappa)} t^{1/4} \int_0^\infty (l_1(x) - 1)\, x^{-3/4}\, dx - O(t)$$

The relationship $l_1(x) - 1 = O(x^{-1})$ as $x \to \infty$ was used here. Integrating by parts in the last integral in (3.5) and applying the formula /8/

$$\frac{d}{dx}\left[e^{-x} x^{c-a} \Phi(a, c; x)\right] = (c - a)\, e^{-x} x^{c-a-1} \Phi(a-1, c; x)$$

$$\Phi\left(\frac{3}{4}, \frac{5}{2}; x\right) = \frac{\Gamma(5/2)}{\Gamma(3/4)\,\Gamma(7/4)} \int_0^1 e^{xu} (1-u)^{3/4} u^{-1/4}\, du$$

we find

$$J_{01}(t) = \frac{2\Gamma(7/4)(\varkappa-1)}{\pi^{1/2}\Gamma(5/2)(\varkappa+1)} t^{1/4} \int_0^\infty e^{-x}\Phi\left(\frac{3}{4}, \frac{5}{2}; x\right)dx + O(t) = \frac{4}{\pi}\Gamma\left(\frac{3}{4}\right)\frac{\varkappa-1}{\varkappa+1} t^{1/4} + O(t)$$

Since L is an even and $h_{11}^{(0)}$ is an odd function of θ, then $J_{02} = 0$.
We use the notation

$$H_j = h_j - h_j^{(0)}, \quad J_{1j} = I_{0j} - J_{0j} = -\int_{U_\delta(O)} (L-1) H_j \, dx$$

We represent J_{1j} as the sum of two integrals, the first of which is extended over the set $\{x\colon 0 < r < t^{1/2}, |\theta| < \pi\}$ and the second over the set $\{x\colon t^{1/2} < r < \delta, |\theta| < \pi\}$. By virtue of the estimate $|H_j| \leqslant cr^{-1/2}$ and the boundedness of the function L the first integral equals $O(t^{3/4})$. We replace the function $L-1$ in the second integral by its asymptotic form

$$-\chi(\tau)\,\mathrm{erfc}\left(r\frac{\sin\tau}{2t^{1/2}}\right) + O\left(\left(\frac{r^2}{4t}\right)^{-N}\right), \quad \frac{r^2}{4t}\to\infty$$

and the function H_j by the sum $H_j^{\pm}(r) + O(\tau r^{-1/2})$, where $H_j^{\pm} = \lim H_j$ as $\theta \to \pm\pi$. Then

$$J_{1j}(t) = \int_{t^{1/2}}^{\delta}\int_0^{\delta} \mathrm{erfc}\left(\frac{r\sin\tau}{2t^{1/2}}\right)d\tau\,(H_j^+ + H_j^-)\,r\,dr + O(t^{3/4})$$

Furthermore, since

$$\int_0^{\delta} \mathrm{erfc}\left(\frac{r\sin\tau}{2t^{1/2}}\right)d\tau = \frac{2}{\sqrt{\pi}r}t^{1/2} - O\left(\frac{t}{r^2}\right)$$

$$J_{1j}(t) = \frac{2t^{1/2}}{\sqrt{\pi}}\int_{\gamma_\delta} h_j\,d\Gamma + O(t^{3/4}), \quad \gamma_\delta = \Gamma \cap U_\delta(O)$$

(the integral is understood in the principal value sense). Therefore

$$I_{01}(t) = \frac{4}{\pi}\Gamma\left(\frac{3}{4}\right)\frac{\varkappa-1}{\varkappa+1}t^{1/4} + \frac{2t^{1/2}}{\sqrt{\pi}}\int_{\gamma_\delta} h_1\,d\Gamma - O(t^{3/4})$$

$$I_{02}(t) = \frac{2t^{1/2}}{\sqrt{\pi}}\int_{\gamma_\delta} h_{11}\,d\Gamma - O(t^{3/4}).$$

The asymptotic form of the integrals I_{Aj}, I_{1j} is found similarly, even somewhat more simply, and has the form

$$I_{Aj} + I_{1j} = \frac{2t^{1/2}}{\sqrt{\pi}}\int_{\Gamma\setminus\gamma_\delta} h_j\,d\Gamma - O(t^{3/4})$$

from which (3.3) follows.

Remark. According to the theorem proved, the sum of the squares of the intensity coefficients $\overline{K}^2 = K_I^2 + K_{II}^2$ grows as $\text{const}\cdot t^{1/2}$ for small t. From the representation of the temperature as an eigenfunction series of the Laplace operator with homogeneous Dirichlet conditions on Γ, it follows that $\overline{K}^2(t) \sim \text{const}\cdot\exp(-2\lambda_1 t)$ for large t, where λ_1 is the first eigenvalue of the Dirichlet problem for the Laplace operator. Since $\overline{K}^2$ is a continuous function of time (see (3.2)), at a certain time it reaches a maximum. If this maximum is sufficiently small, the crack is stable.

According to Theorem 1, for $T_0 > 0$ the coefficient K_I is positive for small t, i.e., tensile stresses originate at the crack tip during cooling of the contour Γ. When $T_0 < 0$ the stresses will be compressive. In particular, the asymptotic form of the time t^* of the beginning of crack propagation

$$t^* \sim \left(\frac{\pi}{4\Gamma(3/4)}\frac{K_{IC}}{\mu m}\frac{1}{T_0}\right)^4, \quad T_0 \gg 1$$

is determined from (3.3).

Here T_0 is the jump in temperature at the crack tip, and K_{IC} is the critical value of the tensile stress intensity coefficient.

4. Taking account of heat transfer.

Let Ω be the same domain as before, and $T(x, t)$ the solution of the equation

$$\Delta T - \sigma^2 T - \frac{\partial T}{\partial t} = 0 \tag{4.1}$$

This equation describes the mean temperature distribution, over the thickness, in a thin plate $\Omega \times [-h, h]$ on whose side surfaces heat transfer from the surrounding zero temperature medium occurs according to Newton's law, $\sigma^2 = k/h$, where k is the coefficient of relative thermal efficiency.

As in Sect.1.1, we assume that the plate had the temperature T_0 at the initial instant, and then its endfaces instantaneously acquired the temperature T_1, i.e.,

$$T|_{t=0} = T_0, \quad T|_\Gamma = T_1 \quad \text{for} \quad t > 0 \tag{4.2}$$

The displacements originating in the plate satisfy the boundary value problem

$$\lambda_* \Delta \mathbf{U} + (\mu + \lambda_*) \operatorname{grad} \operatorname{div} \mathbf{U} = \gamma_* \operatorname{grad} T \quad \text{on} \quad \Omega \tag{4.3}$$

$$\lambda_* \operatorname{div} \mathbf{U} + 2\mu \frac{\partial U_n}{\partial n} = \gamma_* T \quad \text{on} \quad \Gamma$$

$$\mu \left(\frac{\partial U_n}{\partial \tau} + \frac{\partial U_\tau}{\partial n} \right) = 0 \quad \text{on} \quad \Gamma$$

$$(\lambda_* = 2\lambda\mu (\lambda + 2\mu)^{-1}, \ \gamma_* = 2\mu\alpha_T (1 + \nu)(1 - \nu)^{-1})$$

Let F be the solution of the boundary value problem

$$\partial F/\partial t - \Delta F = 0, \ F|_{t=0} = 1, \ F|_\Gamma = 0$$

It is confirmed directly that the function

$$T(x, t) = \exp(-\sigma^2 t)(T_0 - T_1) F(x, t) + T_1 \left(1 - \sigma^2 \int_0^t \exp(-\sigma^2 \tau) F(x, \tau) \, d\tau\right) \tag{4.4}$$

satisfies problem (4.1), (4.2). Let $Q_{\rm I}(t), Q_{\rm II}(t)$ denote the stress intensity coefficients generated by the temperature field F in a plate with zero heat transfer from the external medium. Also let $K_{\rm I}(t), K_{\rm II}(t)$ be stress intensity coefficients in the initial problem. It follows from (3.2) and (4.4) that $(j = \rm I, II)$

$$K_j(t) = \exp(-\sigma^2 t)(T_0 - T_1) Q_j(t) - \sigma^2 T_1 \int_0^t \exp(-\sigma^2 \tau) Q_j(\tau) \, d\tau \tag{4.5}$$

Using Theorem 1, we hence obtain the equality

$$K_{\rm I}(t) = K_{\rm I}^{(0)}(t) + R_{\rm I}(t), \ \text{where}$$

$$K_{\rm I}^{(0)}(t) = \frac{4}{\pi} \Gamma\left(\frac{3}{4}\right) \mu (1 + \nu) \alpha_T \sigma^{-1/2} T_1 S\left(\frac{T_0 - T_1}{T_1}, \sigma^2 t\right)$$

$$S(\Lambda, y) = \Lambda e^{-y} y^{1/4} - \int_0^y e^{-\tau} \tau^{1/4} \, d\tau$$

The residue $R_{\rm I}$ and the intensity coefficient $K_{\rm II}$ allow of the estimate

$$|R_{\rm I}(t)| + |K_{\rm II}(t)| \leqslant c\,(|T_0| + |T_1|) \min\{\sigma^{-1}, \ t^{1/2}\}$$

where c is independent of T_0, T_1, σ, t.

By virtue of (4.5) and (3.3), for $\sigma^2 t \ll 1$ the coefficients $K_{\rm I}, K_{\rm II}$ have the same asymptotic value as in the absence of heat transfer (see (3.3)). The stresses near the crack tip are thereby compressive for $T_1 > T_0$ and tensile for $T_1 < T_0$ for small $\sigma^2 t$.

When $\sigma^2 t \gg 1$, $t \ll 1$ we have

$$K_{\rm I}(t) \sim -\sqrt{2}\, \mu (1 + \nu) \alpha_T \sigma^{-1} T_1$$

and, in particular, the stresses will be tensile (compressive) for $T_1 < 0$ $(T_1 > 0)$.

Let us study the nature of the stresses in the intermediate zone of variation of $\sigma^2 t$ by limiting ourselves to the principal term of the asymptotic form $K_{\rm I}^{(0)}(t)$. Its behaviour depends on the sign of the numbers $T_0, T_1, T_0 - T_1$.

If $T_1 > 0, T_0 < T_1$ or $T_1 < T_0 < 0$ then the function $K_{\rm I}^{(0)}(t)$ varies monotonically between zero and $K_{\rm I}^{(0)}(\infty) = -\sqrt{2}\mu (1 + \nu) \alpha_T \sigma^{-1} T_1$.

If $T_1 < 0, T_0 > 0, T_0 > T_1$, then $K_{\rm I}^{(0)}(t) > 0$ and at the time $t_* = \sigma^{-2}(T_0 - T_1)/(4T_0)$ takes the greatest value

$$K_{\rm I}^{(0)}(t_*) = \frac{4}{\pi} \Gamma\left(\frac{3}{4}\right) \mu (1 + \nu) \sigma^{-1/2} T_1 S\left(\frac{T_0 - T_1}{T_1}, \frac{T_0 - T_1}{4T_0}\right)$$

When $0 < T_1 < T_0$ the function $K_{\rm I}^{(0)}(t)$ is positive in the interval $(0, t_0)$, $t_0 = \sigma^{-2} R((T_0 - T_1)/T_1)$, where $y = R(\Lambda)$ is the single positive root of the equation $S(\Lambda, y) = 0$. For $t > t_0$ it is negative and varies between zero and $K_{\rm I}^{(0)}(\infty)$. The greatest value of the function $K_{\rm I}^{(0)}(t)$ is $K_{\rm I}^{(0)}(t_*)$.

Finally, for $T_0 < T_1 < 0$ the quantity $K_{\rm I}^{(0)}(t)$ is negative in the interval $(0, t_0)$, changes sign at the time t_0, and increases monotonically to $K_{\rm I}^{(0)}(\infty)$.

The following asymptotic formulas are confirmed directly:

$$K_{\rm I}^{(0)}(t_*) \sim \frac{4e^{-1/4}}{\sqrt{2}\pi}\Gamma\left(\frac{3}{4}\right)\mu(1+\nu)\sigma^{-1/4}(T_0-T_1), \quad |T_0| \gg |T_1|$$

$$K_{\rm I}^{(0)}(t_*) \sim \frac{8\sqrt{2}}{5\pi}\Gamma\left(\frac{3}{4}\right)\mu(1+\nu)\sigma^{-1/4}T_1\left(\frac{T_0-T_1}{T_1}\right)^{5/4}, \quad T_0T_1^{-1} \to 1+0$$

sgn T_0	sgn T_1	sgn (T_0-T_1)	Kind of stress	Stability criterion
$\pm$	$+$	$-$	Compressive	The crack is stable
$+$	$-$	$+$	Tensile	$K_{\rm I}^{(0)}(t_*) < K_{\rm IC}$
$-$	$-$	$+$	Tensile	$K_{\rm I}^{(0)}(\infty) < K_{\rm IC}$
$+$	$+$	$+$	Tensile for $t < t_0$, Compressive for $t > t_0$	$K_{\rm I}^{(0)}(t_*) < K_{\rm IC}$
$-$	$-$	$-$	Compressive for $t < t_0$, Tensile for $t > t_0$	$K_{\rm I}^{(0)}(\infty) < K_{\rm IC}$

The time t_0 is an increasing function of the ratio T_0/T_1 such that

$$t_0 \sim \frac{5}{4}\sigma^{-2}\frac{T_0-T_1}{T_1}, \quad T_0T_1^{-1} \to 1+0$$

$$t_0 \sim \sigma^{-2}\log\frac{T_0}{T_1}, \quad T_0T_1^{-1} \to +\infty$$

Deductions from the investigation made on the function $K_{\rm I}^{(0)}(t)$ are collected in the table ($K_{\rm IC}$ is the critical value of the tensile stress intensity coefficient).

REFERENCES

1. PARTON V.Z., Axisymmetric temperature problem for a space with a disc-like crack, PMM, 36, 1, 1972.
2. KUDRYAVTSEV B.A. and PARTON V.Z., Quasistatic temperature problem for a plane with a cut, Problemy Prochnosti, 2, 1970.
3. KUDRYAVTSEV B.A., Quasistationary thermoelasticity problem for a plane with a semi-infinite cut, Dinamika Sploshnoi Sredy. Izd. Inst. Gidrodin. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk 1970.
4. POBEREZHNYI O.V. and GAIVAS', I.V., Influence of a non-stationary temperature field and thermal efficiency of a plate on the stress intensity coefficients. Prikl. Mekhan., 18, 6, 1982.
5. POBEREZHNYI O.V., On the influence of the magnitude of the domain of temperature load action on the stress intensity coefficients of a plate with a semi-infinite cut. Matematicheskie Metody i Fiziko-mekhanicheskie Polia. Naukova Dumka, Kiev, 1983.
6. PARTON V.Z. and MOROZOV E.M., Mechanics of Elastic-Plastic Fracture. Nauka, Moscow, 1974.
7. MAZ'IA V.G. and PLAMENEVSKII B.A., On the coefficients in the asymptotic form of the solution of elliptic boundary value problems in domains with conic points. Math. Nachr., 76, 1977.
8. BATEMAN H. and ERDELYI A., Higher Transcendental Functions /Russian translation/ Nauka, Moscow, 1965.

Translated by M.D.F.